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LETTER TO THE EDITOR

The critical point of fractal percolation in three and more dimensions

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Abstract. The fractal percolation model is constructed by the progressive replacement of each cube by a random collection of sub-cubes, each sub-cube being retained with probability p. There is a critical probability $p_c(M, d)$ depending on the dimension d and the substitution ratio M. We point out that, for large M, the value of $p_c(M, d)$ is near to that of site percolation on a lattice derived from the d-dimensional hypercubic lattice by the addition of certain bonds.

Let $d \geqslant 2$, $M \geqslant 2$ and $0 < p < 1$. We consider a 'random fractal' C constructed as follows. Let C_0 be the unit cube $[0, 1]^d$ of $\mathbb{R}^d$. Divide C_0 into M^d equal closed cubes each of side-length M^{-1} in the obvious way. We select each of these cubes with probability p independently of all other cubes, and denote by C_1 the union of the cubes thus selected. In the same way, we divide each cube of C_1 into M^d sub-cubes of side M^{-2}, select each of these with probability p independently of the other sub-cubes, and let C_2 be the union of the selected sub-cubes of C_1. Continuing in this way, we obtain a decreasing sequence of closed sets $C_0 \supset C_1 \supset C_2 \supset \ldots$, where C_n is the union of cubes of side-length M^{-n}. Let $C = \bigcap_{n=0}^{\infty} C_n$ be the 'limit' of this sequence of sets. Provided that this process does not become extinct (i.e. provided that $C_n \neq \varnothing$ for all n), the limit set C may be thought of as a 'random fractal'. The dimension of C is given by

$$\dim_H C = \dim_B C = d - \frac{|\log p|}{\log M} \tag{1}$$

with probability 1, where $\dim_H$ and $\dim_B$ denote Hausdorff and box dimensions; see Falconer (1990) for details and references concerning fractals.

Constructions of this form have been used to model a variety of physical processes, such as the distribution of galaxies in the universe, or intermittency in turbulence; see Mandelbrot (1983, section 23).

We consider the topology of the random set C. If $0 \leqslant p < M^{1-d}$, then by (1) almost surely $\dim_H C < 1$, implying that C is totally disconnected. On the other hand, if p is sufficiently close to 1, then there is a positive probability that C contains 'long paths'; that is to say, there is a positive probability that opposite faces of C_0 are joined by a path contained within C, in which case we say that *percolation occurs*. It is actually the case that there exists a critical probability $p_c(M, d)$ such that, if $p < p_c(M, d)$ then C is totally disconnected almost surely, and if $p > p_c(M, d)$ then percolation occurs in C with positive probability. (In fact, at least if $d = 2$, percolation occurs if $p \geqslant p_c(M, d)$; see Chayes *et al* 1988.)

It is natural to relate fractal percolation to discrete lattice percolation, described fully in Grimmett (1989). Consider the lattice graph $\mathbb{Z}^d$, with vertices at the points with integer coordinates, and an edge joining $x, y \in \mathbb{Z}^d$ if and only if the Euclidean distance from x and y satisfies $|x-y|=1$. Let $0<p<1$ and declare each edge of the lattice *open* with probability p, independently of the other edges. Let S be the set of open edges in the lattice. Then there exists a critical probability $p_c(\mathbb{Z}^d)$ such that if $p<p_c(\mathbb{Z}^d)$ then almost surely S has no unbounded connected component, whereas if $p>p_c(\mathbb{Z}^d)$ then with probability one S has a unique unbounded connected component. It may be shown that if B is a very large lattice cube in $\mathbb{Z}^d$, then the probability of opposite faces of B being joined by a path in S is very small if $p<p_c(\mathbb{Z}^d)$, and is close to one if $p>p_c(\mathbb{Z}^d)$. It is natural to try to relate $p_c(M, d)$ for large M to $p_c(\mathbb{Z}^d)$. Chayes and Chayes (1989) showed that $\lim_{M\to\infty} p_c(M, 2)=p_c(\mathbb{Z}^2)$. The purpose of this letter is to point out that if $d \geqslant 3$ then the limit of $p_c(M, d)$ exists as $M\to\infty$, the limiting value being rather smaller than $p_c(\mathbb{Z}^d)$, being equal to the critical probability of a certain lattice $\mathbb{L}^d$ derived from $\mathbb{Z}^d$ by the addition of new edges.

For $d \geqslant 2$ we construct the lattice graph $\mathbb{L}^d$ as follows: the vertices of $\mathbb{L}^d$ are the points of $\mathbb{Z}^d$, i.e. the points in d-dimensional Euclidean space with integer coordinates; two vertices $x=(x_1, \ldots, x_d)$ and $y=(y_1, \ldots, y_d)$ are joined by an edge if and only if $|x_i-y_i| \leqslant 1$ for all i *and in addition* $x_i=y_i$ for some i satisfying $1 \leqslant i \leqslant d$. Thus x and y are joined by an edge in $\mathbb{L}^d$ if and only if the coordinate cubes of side 1 centred at x and y intersect in a set of dimension at least 1, i.e. in a line segment, square, three-dimensional cube, etc. Note that x and y are not joined if these cubes have just a single point in common.

Just as with the lattice $\mathbb{Z}^d$, we declare the edges of $\mathbb{L}^d$ open with probability p independently, and let S denote the set of open edges. There is a critical probability $p_c(\mathbb{L}^d)$ such that if $p<p_c(\mathbb{L}^d)$ then S has no unbounded component, but if $p>p_c(\mathbb{L}^d)$ *then S has an unbounded component.*

Of course, the lattices $\mathbb{L}^2$ and $\mathbb{Z}^2$ are identical, but $\mathbb{Z}^d$ is a proper subgraph of $\mathbb{L}^d$ for $d \geqslant 3$. (Each vertex of $\mathbb{Z}^d$ has degree 2^d, but each vertex of $\mathbb{L}^d$ has degree 3^d-2^d-1.)

We claim that

$$\lim_{M\to\infty} p_c(M, d)=p_c(\mathbb{L}^d). \tag{2}$$

For $d \geqslant 3$, the value of this limit is strictly less than the $p_c(\mathbb{Z}^d)$ that might have been expected.

The detailed proof of (2), given in Falconer and Grimmett (1991), is long and makes use of some results from percolation theory, in particular the recent results of Grimmett and Marstrand (1990), relating percolation in the whole of $\mathbb{Z}^d$ and in a slab. Here we indicate why it is the lattice graph $\mathbb{L}^d$ that must be considered rather than $\mathbb{Z}^d$, and we discuss the values of the critical probabilities.

We say that percolation occurs in C_n if there is a path inside C_n joining two specified opposite faces of C_0. We say that *edge-percolation* occurs in C_n if these faces of C_0 are connected by a sequence or cubes in C_n, consecutive cubes having at least a (one-dimensional) edge in common—thus single point contact between cubes is excluded in the case of edge percolation. It is easy to see that percolation occurs in C if and only if percolation occurs in C_n for all $n \in \mathbb{Z}^+$. However, we claim that

$$P(\text{percolation in } C)=\lim_{n\to\infty} P(\text{edge-percolation in } C_n).$$

To see this, note that if percolation occurs in C but no edge percolation occurs in C_n, then every sequence of cubes $Q_1, \ldots, Q_k$ in C_n that connects opposite faces of C_0 has at least one consecutive pair, Q_i, Q_{i+1}, say, with a single point x in common. For there to be a path in C joining opposite faces of C_0 inside this sequence of cubes, C_m must include the cubes of sides M^{-m} inside Q_i and Q_{i+1} which contain the point x, for all $m \geq n$. The probability of this happening for a particular sequence of cubes $Q_1, \ldots, Q_k$ is zero; since there are finitely many possible pairs of cubes in C_n with a single point in common,

$$P(\text{percolation in } C, \text{ percolation but no edge percolation in } C_n) = 0$$

for all n. Consequently

$$P(\text{percolation in } C) = \lim_{n\to\infty} P(\text{percolation in } C_n)$$

$$= \lim_{n\to\infty} P(\text{edge-percolation in } C_n). \qquad (3)$$

(An argument in the plane related to that which leads to (3) appears in Chayes *et al* (1988).)

A lengthy argument shows that the limit in (3) can be made as close to P(edge-percolation in C_1) as we wish by choosing M sufficiently large. There is an obvious correspondence between edge-percolation in C_1 and the existence of an open path between opposite faces of a d-dimensional box of side $M-1$ in the lattice graph $\mathbb{L}^d$. It may be shown that the critical probability for the existence of such an open path converges to $p_c(\mathbb{L}^d)$ as $M \to \infty$.

Estimates for $p_c(M, d)$ are of interest, particularly for $d=2$ or 3 and M large. However, little is known about these values; it is not even certain that $p_c(M, d)$ decreases monotonically as M increases. In the plane, Chayes *et al* (1988) have shown that $M^{-1/2} \leq p_c(M, 2)$ and $p_c(3, 2) < 0.999$; numerical simulation suggests that $p_c(3, 2) \simeq 0.85$.

For fractal percolation in three dimensions, we have indicated that $p_c(M, 3) \to p_c(\mathbb{L}^3)$ as $M \to \infty$. Since $\mathbb{L}^3$ is obtained from $\mathbb{Z}^3$ by adding edges in a regular fashion, it follows from results of Aizenman and Grimmett (1991) that $p_c(\mathbb{L}^3) < p_c(\mathbb{Z}^3)$. The following argument provides a quantitative demonstration of this.

Consider the following subgraph of $\mathbb{L}^3$. Let

$$V = \{(x, y, z) \in \mathbb{Z}^3 \colon x + y \text{ is even}\}.$$

Let E be the set of edges in $\mathbb{L}^3$ that join pairs of points of V. Then (V, E) is a subgraph of $\mathbb{L}^3$ which is isomorphic to the lattice graph $\mathbb{Z}^3$. With each edge $e \in E$, we may associate a pair of edges $a(e), b(e) \in \mathbb{L}^3$ such that e, $a(e)$, $b(e)$ form a triangle and such that the following property holds: if e_1 and e_2 are distinct edges of E then e_1, $a(e_1)$, $b(e_1)$, e_2, $a(e_2)$, $b(e_2)$ are distinct edges of $\mathbb{L}^3$.

One way of achieving this is as follows. If e joins $(x, y, z) \in V$ to $(x+r, y+s, z)$ where $r, s = \pm 1$, and (without loss of generality) x, y are even, we take the common vertex of $a(e)$ and $b(e)$ to be $(x + \frac{1}{2}(r+s), y - \frac{1}{2}(r-s), z+1)$. If e joins $(x, y, z) \in V$ to $(x, y, z+1)$ then we take the common vertex to be $(x+1, y, z)$.

Now consider the random graph obtained by selecting each edge of $\mathbb{L}^3$ with probability p independently of other edges. If e is an edge of (V, E), there is a probability p that this edge itself is open in $\mathbb{L}^3$ and there is a probability p^2 that both $a(e)$ and $b(e)$ are open in $\mathbb{L}^3$. Thus there is a probability of

$$1 - (1-p)(1-p^2) = p + p^2 - p^3$$

that the end-vertices of e are joined either by the edge e itself or by the two edges $a(e)$ and $b(e)$. These events are independent, for $e \in E$. It follows that $\mathbb{L}^3$ has an unbounded infinite component if $p+p^2-p^3$ exceeds the critical probability for percolation on $\mathbb{Z}^3$. Thus

$$p+p^2-p^3 \leqslant p_c(\mathbb{Z}^3)$$

where $p=p_c(\mathbb{L}^3)$. Numerical evidence suggests that $p_c(\mathbb{Z}^3) \simeq 0.25$ (see Essam 1972, p 224); using this figure it follows that $\lim_{M\to\infty} p_c(M,3) = p_c(\mathbb{L}^3) \leqslant 0.214$.

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